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Polynomial Functions

Definition. A **polynomial in x** is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer and each coefficient a_i , $i \in \{0, 1, 2, \dots, n\}$, is a complex number.

Definition. If f of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer and each coefficient is a complex number, then f is called a **polynomial function**.

Fundamental Theorem of Algebra. A polynomial function of degree n , $n \geq 1$, has at least one and at most n complex zeros, and the sum of the multiplicities of the zeros is exactly n .

Division Algorithm. Given an arbitrary polynomial $f(x)$ of degree m and a non-zero polynomial $d(x)$ of degree n , where $m \geq n$, there is exactly one $q(x)$ and $r(x)$ such that, for all x , $f(x) = d(x) \cdot q(x) + r(x)$ and the degree of $r(x)$ is less than the degree of $d(x)$. $q(x)$ and $r(x)$ are called the **quotient** and **remainder**. The degree of the quotient $q(x)$ is $m - n$.

Remainder Theorem. If a polynomial $f(x)$ is divided by $x - \alpha$ and the remainder is R , then $R = f(\alpha)$.

Factor Theorem. A polynomial $f(x)$ has a factor $x - \alpha$ if and only if $f(\alpha) = 0$.

Rational Root Theorem. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial over the integers. If p, q are relatively prime integers and $\frac{p}{q}$ is a root of $f(x) = 0$, then $q \mid a_n$ and $p \mid a_0$.

Corollary. If $f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial over the integers, then any rational root of $f(x) = 0$ is an integer which is a divisor of a_0 .

■ Notes

We say that an integer a **divides** integer b if there is an integer c such that $b = ac$. (The fact that an integer a divides an integer b is symbolized by $a \mid b$ (pronounced " a divides b "). When a divides b , we say a is a **divisor** of b . We say that two integers are **relatively prime** if they have no common divisors other than 1 and -1 .

When the coefficients of a polynomial are restricted to the set of rational numbers, we say the polynomial is a polynomial over the rational numbers. Similarly, a polynomial whose coefficients are restricted to integers is called a polynomial over the integers.